## B.Sc. IV SEMESTER

# Mathematics 

## PAPER - I

## VECTOR CALCULUS

AND INFINITE SERIES

## UNIT-V

## INFINITE SERIES - III

Syllabus:

## Unit - V

## Infinite Series - III

Absolute Convergence and Conditional Convergence of series. Alternating series, Leibnitz theorem,

### 5.1. Introduction:

All of the series convergence tests we have used require that the underlying sequence $\left\{a_{n}\right\}$ be a positive sequence. In this Unit we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative.

### 5.2. Alternating Series:

Definition: A series of the form $\sum_{n=1}^{\infty}(-1)^{n-1} u_{n}=u_{1}-u_{2}+u_{3}-u_{4}+u_{5}-\ldots . . . .$. where $u_{n}>0, \forall n$ is called an alternating series.

### 5.3. Leibnitz's test for Alternating Series:

Statement: The alternating series

$$
\begin{aligned}
& \sum_{n=1}^{\infty}(-1)^{n-1} u_{n}=u_{1}-u_{2}+u_{3}-u_{4}+u_{5}-\ldots \ldots \ldots . . \quad\left(u_{n}>0, \forall n\right) \text { converges if } \\
& \text { i. } u_{n} \geq u_{n+1}, \forall n \quad \text { and } \\
& \text { ii. } \lim _{n \rightarrow \infty} u_{n}=0
\end{aligned}
$$

Proof: Let $S_{n}$ denote the $n^{t h}$ partial sum of the series $\sum_{n=1}^{\infty}(-1)^{n-1} u_{n}$

$$
\begin{aligned}
\therefore S_{2 n}=u_{1}-u_{2} & +u_{3}-u_{4}+u_{5}-\ldots \ldots \ldots . .-u_{2 n-2}+u_{2 n-1}-u_{2 n} \\
& =u_{1}-\left(u_{2}-u_{3}\right)-\left(u_{4}-u_{5}\right)-\ldots \ldots \ldots . .\left(u_{2 n-2}-u_{2 n-1}\right)-u_{2 n} \\
& =u_{1}-\left\{\left(u_{2}-u_{3}\right)+\left(u_{4}-u_{5}\right)+\ldots \ldots \ldots .\left(u_{2 n-2}-u_{2 n-1}\right)+u_{2 n}\right\} \\
& \leq u_{1}\left(\because u_{n} \geq u_{n+1} \& u_{n}>0, \forall n\right) \\
\therefore S_{2 n} & \leq u_{1}, \forall n \\
\Rightarrow & \text { The sequence }\left\langle S_{2 n}\right\rangle \text { is bounded above. }
\end{aligned}
$$

Also

$$
\begin{aligned}
& S_{2 n+2}=S_{2 n}+u_{2 n+1}-u_{2 n+2} \\
& \Rightarrow S_{2 n+2}-S_{2 n}=u_{2 n+1}-u_{2 n+2} \geq 0\left(\because u_{n} \geq u_{n+1}, \forall n\right) \\
& \quad \therefore S_{2 n+2}-S_{2 n} \geq 0 \Rightarrow S_{2 n+2} \geq S_{2 n}, \forall n
\end{aligned}
$$

$\Rightarrow$ The sequence $\left\langle S_{2 n}\right\rangle$ is monotonically increasing.

Since, every monotonically increasing sequence which is bounded above converges, therefore the sequence $\left\langle S_{2 n}\right\rangle$ converges.
Let the sequence $\left\langle S_{2 n}\right\rangle$ converges to $S$ i.e. $\lim _{n \rightarrow \infty} S_{2 n}=S$.
Now,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} S_{2 n+1}=\lim _{n \rightarrow \infty}\left(S_{2 n}+u_{2 n+1}\right)=\lim _{n \rightarrow \infty} S_{2 n}+\lim _{n \rightarrow \infty} u_{2 n+1}=S+0=S\left(\because \lim _{n \rightarrow \infty} u_{n}=0\right) \\
& \therefore \lim _{n \rightarrow \infty} S_{2 n+1}=S
\end{aligned}
$$

$\therefore$ The sequences $\left\langle S_{2 n}\right\rangle$ and $\left\langle S_{2 n+1}\right\rangle$ converges to the same number $S$. $\Rightarrow \forall \in>0$ there exist positive integers $m_{1} \& m_{2}$ such that

$$
\left|S_{2 n}-S\right| \leq \frac{\epsilon}{2}, \forall 2 n>m_{1} \quad \& \quad\left|S_{2 n+1}-S\right| \leq \frac{\epsilon}{2}, \forall 2 n+1>m_{2}
$$

Let $\quad m=\max \left(m_{1}, m_{2}\right)$

$$
\begin{aligned}
& \therefore\left|S_{2 n}-S\right| \leq \frac{\epsilon}{2}, \forall 2 n>m \&\left|S_{2 n+1}-S\right| \leq \frac{\epsilon}{2}, \forall 2 n+1>m \\
& \Rightarrow\left|S_{n}-S\right| \leq \frac{\epsilon}{2}<\epsilon, \forall n>m \\
& \quad \Rightarrow \text { The sequence }\left\langle S_{n}\right\rangle \text { is converges to } S .
\end{aligned}
$$

$$
\therefore \text { The given series is convergent. }
$$

## Example : Discuss the convergence of the following series

1. $\sum \frac{(-1)^{n-1} n}{5^{n}}$
2. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$ $\qquad$
3. $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+$ $\qquad$
4. $\frac{1}{2^{3}}-\frac{1}{3^{3}}(1+2)+\frac{1}{4^{3}}(1+2+3)-$ $\qquad$

## Solution:

1. Which is an alternating series \& here $u_{n}=\frac{n}{5^{n}}, \therefore u_{n+1}=\frac{n+1}{5^{n+1}}$

Consider

$$
u_{n}-u_{n+1}=\frac{n}{5^{n}}-\frac{n+1}{5^{n+1}}=\frac{1}{5^{n}}\left(n-\frac{n+1}{5}\right)=\frac{1}{5^{n}}\left(\frac{5 n-(n+1)}{5}\right)
$$

$$
\begin{aligned}
= & \frac{1}{5^{n}}\left(\frac{5 n-n-1}{5}\right)=\frac{1}{5^{n+1}}\left(\frac{4 n-1}{5}\right) \geq 0, \quad \forall n \\
\Rightarrow & u_{n}-u_{n+1} \geq 0 \Rightarrow u_{n} \geq u_{n+1}, \forall n \quad \text { and } \\
& \lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} \frac{n}{5^{n}}=0 \\
\therefore & u_{n} \geq u_{n+1}, \forall n \& \lim _{n \rightarrow \infty} u_{n}=0
\end{aligned}
$$

By Leibnitz's test, the given series is convergent.
2. Which is an alternating series \& $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \ldots=\sum \frac{(-1)^{n-1}}{2 n-1}$ Here $u_{n}=\frac{1}{2 n-1}, \therefore u_{n+1}=\frac{1}{2 n+1}$

Consider

$$
\begin{aligned}
u_{n}- & u_{n+1}=\frac{1}{2 n-1}-\frac{1}{2 n+1}=\frac{2 n+1-(2 n-1)}{(2 n+1)(2 n-1)} \\
= & \frac{2 n+1-2 n+1}{(2 n+1)(2 n-1)}=\frac{2}{(2 n+1)(2 n-1)} \geq 0, \forall n \\
\Rightarrow & u_{n}-u_{n+1} \geq 0 \Rightarrow u_{n} \geq u_{n+1}, \forall n \quad \text { and } \\
& \lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} \frac{1}{2 n-1}=0 \\
\therefore & u_{n} \geq u_{n+1}, \forall n \& \lim _{n \rightarrow \infty} u_{n}=0
\end{aligned}
$$

By Leibnitz's test, the given series is convergent.
3. Which an alternating series \&
$\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\ldots \ldots \ldots=\sum \frac{(-1)^{n-1}}{\log n}, \forall n \geq 2$
Here $u_{n}=\frac{1}{\log n}, n \geq 2, \therefore u_{n+1}=\frac{1}{\log (n+1)}$
Consider
$u_{n}-u_{n+1}=\frac{1}{\log n}-\frac{1}{\log (n+1)} \geq 0, \forall n$

$$
\left(\begin{array}{l}
\because n \leq n+1, \forall n \Rightarrow \log n \leq \log (n+1) \Rightarrow \frac{1}{\log n} \geq \frac{1}{\log (n+1)} \\
\Rightarrow \\
\quad \frac{1}{\log n}-\frac{1}{\log (n+1)} \geq 0 \\
\quad \Rightarrow u_{n}-u_{n+1} \geq 0 \Rightarrow u_{n} \geq u_{n+1}, \forall n \text { and } \\
\quad \lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} \frac{1}{\log n}=0 \\
\quad \therefore u_{n} \geq u_{n+1}, \forall n \& \lim _{n \rightarrow \infty} u_{n}=0
\end{array}\right.
$$

By Leibnitz's test, the given series is convergent.
4. Which is an alternating series \&

$$
\begin{aligned}
& \frac{1}{2^{3}}-\frac{1}{3^{3}}(1+2)+\frac{1}{4^{3}}(1+2+3)-\ldots \ldots=\sum(-1)^{n-1} \frac{1}{(n+1)^{3}}(1+2+3+\ldots+n) \\
& u_{n}=\frac{1}{(n+1)^{3}}(1+2+3+\ldots+n)=\frac{1}{(n+1)^{3}}\left(\frac{n(n+1)}{2}\right)=\frac{n}{2(n+1)^{2}}, \\
& \therefore u_{n+1}=\frac{n+1}{2(n+2)^{2}}
\end{aligned}
$$

Consider

$$
\begin{aligned}
u_{n}-u_{n+1} & =\frac{n}{2(n+1)^{2}}-\frac{n+1}{2(n+2)^{2}}=\frac{1}{2}\left(\frac{n}{(n+1)^{2}}-\frac{n+1}{(n+2)^{2}}\right) \\
& =\frac{1}{2}\left(\frac{n(n+2)^{2}-(n+1)^{2}(n+1)}{(n+1)^{2}(n+2)^{2}}\right)=\frac{1}{2}\left(\frac{n(n+2)^{2}-(n+1)^{3}}{(n+1)^{2}(n+2)^{2}}\right) \\
& =\frac{1}{2}\left(\frac{n\left(n^{2}+4 n+4\right)-\left(n^{3}+3 n^{2}+3 n+1\right)}{(n+1)^{2}(n+2)^{2}}\right) \\
& =\frac{1}{2}\left(\frac{n^{3}+4 n^{2}+4 n-n^{3}-3 n^{2}-3 n-1}{(n+1)^{2}(n+2)^{2}}\right) \\
& =\frac{1}{2}\left(\frac{n^{2}+n-1}{(n+1)^{2}(n+2)^{2}}\right) \geq 0, \forall n \\
\Rightarrow u_{n} & -u_{n+1} \geq 0 \Rightarrow u_{n} \geq u_{n+1}, \forall n \quad \text { and }
\end{aligned}
$$

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} \frac{n}{2(n+1)^{2}}=\frac{1}{2} \lim _{n \rightarrow \infty} \frac{n}{n^{2}\left(1+\frac{1}{n}\right)^{2}} \\
& =\frac{1}{2} \lim _{n \rightarrow \infty} \frac{1}{n\left(1+\frac{1}{n}\right)^{2}}=\frac{1}{2} \times 0=0 \\
& \therefore u_{n} \geq u_{n+1}, \forall n \& \lim _{n \rightarrow \infty} u_{n}=0
\end{aligned}
$$

By Leibnitz's test, the given series is convergent.

### 5.4.Absolute Convergence:

Definition: A series $\sum_{n=1}^{\infty} u_{n}$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty}\left|(-1)^{n-1} u_{n}\right|$ i.e. $\sum_{n=1}^{\infty}\left|u_{n}\right|$ is convergent.

### 5.5. Conditional Convergence:

Definition: A series $\sum_{n=1}^{\infty} u_{n}$ is said to be conditional convergent if the series $\sum_{n=1}^{\infty}\left|(-1)^{n-1} u_{n}\right|$ i.e. $\sum_{n=1}^{\infty}\left|u_{n}\right|$ is divergent.
5.6. Theorem: Every absolutely convergent series is convergent, but converse is not true.
Proof: Since $\sum_{n=1}^{\infty} u_{n}$ is absolutely convergent series.

$$
\Rightarrow \sum_{n=1}^{\infty}\left|u_{n}\right| \text { is convergent. }
$$

$\therefore$ By Cauchy's general principle of convergence
$\Rightarrow \forall \in>0$ there exist positive integer $m$ such that

$$
\begin{align*}
& \left|\left|u_{m+1}\right|+\left|u_{m+2}\right|+\left|u_{m+3}\right|+\ldots \ldots . .+\left|u_{n}\right|\right|<\epsilon, \forall n \geq m \\
& \left|u_{m+1}\right|+\left|u_{m+2}\right|+\left|u_{m+3}\right|+\ldots \ldots . .+\left|u_{n}\right|<\epsilon, \forall n \geq m \tag{1}
\end{align*}
$$

Consider

$$
\begin{gathered}
\left|u_{m+1}+u_{m+2}+u_{m+2}+\ldots \ldots .+u_{n}\right| \leq \\
\quad\left|u_{m+1}\right|+\left|u_{m+2}\right|+\left|u_{m+3}\right|+\ldots \ldots \ldots+\left|u_{n}\right| \\
(\because|x+y| \leq|x|+|y|) \\
\Rightarrow\left|u_{m+1}+u_{m+2}+u_{m+2}+\ldots+u_{n}\right| \leq \in, \forall n \geq m \quad(\because \text { From Eq.(1) })
\end{gathered}
$$

$\therefore$ By Cauchy's general principle of convergence, the series $\sum_{n=1}^{\infty} u_{n}$ is convergent.
Hence, $\sum_{n=1}^{\infty} u_{n}$ is absolutely convergent series $\Rightarrow \sum_{n=1}^{\infty} u_{n}$ is convergent.

## Converse is not true:

i.e. Every convergent series is need not be absolutely convergent.

Consider an example:

$$
\sum \frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots \ldots
$$

$$
\text { Here } u_{n}=\frac{1}{n}, \therefore u_{n+1}=\frac{1}{n+1}
$$

Consider

$$
\begin{aligned}
& u_{n}-u_{n+1}=\frac{1}{n}-\frac{1}{n+1}=\frac{n+1-n}{n(n+1)}=\frac{1}{n(n+1)} \geq 0, \forall n \\
& \Rightarrow u_{n}-u_{n+1} \geq 0 \Rightarrow u_{n} \geq u_{n+1}, \forall n \text { and } \\
& \lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} \frac{1}{n}=0 \\
& \therefore u_{n} \geq u_{n+1}, \forall n \& \lim _{n \rightarrow \infty} u_{n}=0
\end{aligned}
$$

By Leibnitz's test, the series $\sum \frac{(-1)^{n-1}}{n}$ is convergent.
But, the series $\sum\left|\frac{(-1)^{n-1}}{n}\right|=\sum \frac{1}{n}$ is divergent. $(\because$ It is $\mathrm{p}-$ series with $\mathrm{p}=1)$.

Example 1: Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt[3]{n^{2}}}$ converges absolutely, conditionally, or not at all.

Solution: First we check absolute convergence $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt[3]{n^{2}}}$.

$$
\sum_{n=1}^{\infty}\left|\frac{(-1)^{n}}{\sqrt[3]{n^{2}}}\right|=\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^{2}}} \quad \text { is a p-series with } p \geq \frac{2}{3}
$$

So the series of absolute values diverges. The original series is not absolutely convergent.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_{n}=\frac{1}{n^{2 / 3}}$.
Check the two conditions.

1. $\lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} \frac{1}{n^{2 / 3}}=0$.
2. Further $a_{n+1} \leq a_{n}$ because $\frac{1}{(n+1)^{2 / 3}}<\frac{1}{n^{2 / 3}}$.

Since the two conditions of the alternating series test are satisfied, $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt[3]{n^{2}}}$ is conditionally convergent by the alternating series test.

Example 2: Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^{n}}{\sqrt{n^{2}-1}}$ converges absolutely, conditionally, or not at all.
Solution: First we check absolute convergence . $\sum_{n=2}^{\infty}\left|\frac{(-1)^{n}}{\sqrt{n^{2}-1}}\right|=\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^{2}-1}}$.
So let's use the limit comparison test. The terms of the series are positive and $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=\lim _{n \rightarrow \infty} \frac{1}{\sqrt{n^{2}-1}}=1>0$.
Since the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series with $\mathrm{p}=1$ ), then $\sum_{n=2}^{\infty}\left|\frac{(-1)^{n}}{\sqrt{n^{2}-1}}\right|$ diverges by the limit comparison test. So the series does not converge absolutely.

Since the series is alternating and not absolutely convergent, we check for conditional convergence using the alternating series test with $a_{n}=\frac{1}{\sqrt{n^{2}-1}}$.
Check the two conditions, 1. $\lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} a_{n} \frac{1}{\sqrt{n^{2}-1}}=0$.
2. Further $a_{n+1} \leq a_{n}$ is decreasing because $\frac{1}{\sqrt{(n+1)^{2}-1}}<\frac{1}{\sqrt{n^{2}-1}}$. (You could also show the derivative is negative.)
Since the two conditions of the alternating series test are satisfied, $\sum_{n=2}^{\infty} \frac{(-1)^{n}}{\sqrt{n^{2}-1}}$ is conditionally convergent by the alternating series test.

Example 3: Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^{n}\left(2 n^{4}+7\right)}{6 n^{9}-2 n}$ converges absolutely, conditionally, or not at all.
Solution: First we check absolute convergence $. \sum_{n=1}^{\infty}\left|\frac{(-1)^{n}\left(2 n^{4}+7\right)}{6 n^{9}-2 n}\right|=\sum_{n=2}^{\infty} \frac{\left(2 n^{4}+7\right)}{6 n^{9}-2 n}$. So let's use the limit comparison test. The terms of the series are positive and $\lim _{n \rightarrow \infty} \frac{a_{n}}{b_{n}}=\lim _{n \rightarrow \infty} \frac{\left(2 n^{4}+7\right)}{6 n^{9}-2 n}=\frac{1}{3}>0$.
Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^{5}}$ diverges ( p -series with $\mathrm{p}=5>1$ ), then $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n}\left(2 n^{4}+7\right)}{6 n^{9}-2 n}\right|$ converges by the limit comparison test. So the series converges absolutely.

### 5.7. Uniform Convergence:

Consider the series $\sum_{n=1}^{\infty} u_{n}(x)$ whose terms depend upon a variable $x$, other than the running $n$. The series may converge for some value of $x$ and diverge for others. Suppose the series converges for all $x$ in some closed interval $a \leq x \leq b$.

The sum of the series will be, in general $a$ function of $x$ i.e.

$$
s(x)=\sum_{n=1}^{\infty} u_{n}(x), a \leq x \leq b .
$$

The $n^{t h}$ partial sum of the series is

$$
s_{n}(x)=\sum_{k=1}^{n} u_{k}(x), a \leq x \leq b
$$

Definition: A sequence of functions $\left\langle s_{n}(x)\right\rangle$ is uniformly convergent to the function $s(x)$ over the closed interval $[a, b]$ if $\forall \in>0$ there exists $n_{0}(\in)$ such that

$$
\left|s_{n}(x)-s(x)\right|<\in, \forall n>n_{0}(\in) \& \forall x \in[a, b] .
$$

5.8. Theorem 4. (Weierstrass's M-test): A series of functions $\sum f_{n}$ will converge uniformly (and absolutely) on [a, b] if there exists a convergent series $\sum M_{n}$ of positive numbers such that for all $x \in[a, b]$ such that $\left|f_{n}(x)\right| \leq M_{n}$, for all $n$.

Examples: Test the following series for uniform convergence of

1. $\sum \frac{\cos n x}{n^{p}}$
2. $\sum \frac{1}{n^{p}+n^{q} x^{2}}$
3. $\sum \frac{1}{n^{p}+n^{4} x^{2}}$

Solution:1. Given Series is $\sum \frac{\cos n x}{n^{p}}$

$$
\begin{aligned}
& \quad\left|\frac{\cos n x}{n^{p}}\right|=\frac{|\cos n x|}{n^{p}} \\
& \therefore \quad \frac{|\cos n x|}{n^{p}} \leq \frac{1}{n^{p}}\left(=M_{n}\right) \text { for all values of } x \\
& \text { Since } \quad \sum_{n=1}^{\infty} M_{n}=\sum_{n=1}^{\infty} \frac{1}{n^{p}}, \text { converges } \mathrm{p}>1
\end{aligned}
$$

By M test, the given series converges for all real values of $\mathrm{x} p>1$.

Examples 2: Test the convergence of the series $1+x+x^{2}+\ldots \ldots . . . . . . x^{n-1}+\ldots . . . .$. for uniform convergence in the interval $\left(\frac{-1}{2}, \frac{1}{2}\right)$.
Solution:( Please Try yourself).

## Exercise

1. Discuss the convergence of the following series,
a) $\sum_{n \rightarrow 1}^{\infty} \frac{(-1)^{n+1}}{n}$
b) $\sum_{n \rightarrow 1}^{\infty} \frac{(-1)^{n}}{\sqrt{3 n+1}}$
c) $\sum_{n \rightarrow 1}^{\infty} \frac{(-1)^{n}}{n!}$
d) $\sum_{n \rightarrow 1}^{\infty} \frac{(-1)^{n}}{(2 n-1)!}$
e) $\sum_{n \rightarrow 1}^{\infty}(-1)^{n} \frac{2^{n}+1}{3^{n}-2}$
f) $\sum_{n \rightarrow 1}^{\infty}(-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \ldots \ldots \ldots(2 n-1)}{1 \cdot 4 \cdot 7 \ldots \ldots \ldots(3 n-2)}$
g) $\quad \sum_{n \rightarrow 1}^{\infty} \frac{(-1)^{n} n!}{1 \cdot 3 \cdot 5 \ldots \ldots \ldots(2 n-1)}$
